Int. J. Heat Mass Transfer. Vol. 22, pp. 777–781
© Pergamon Press Ltd. 1979. Printed in Great Britain

0017-9310/79/0501-0777 $02.00/0

PERIODIC HEAT TRANSFER WITH TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

M. S. Sodha, I. C. Goyal, S. C. Kaushik, G. N. Tiwari, A. K. Seth
Physics Department and Centre of Energy Studies, Indian Institute of Technology, New Delhi-110029, India

and

M. A. S. Malik
Engineering Division, Kuwait Institute of Scientific Research, P.O. Box 12009, Kuwait

(Received 17 November 1977 and in revised form 16 August 1978)

NOMENCLATURE

A_0, mean value of the surface temperature [°C];
A_n, amplitude of the nth harmonic of the surface temperature [°C];
c, specific heat of the medium [kJ/kg °C];
i, $= (-1)^{1/2}$;
K_0, thermal conductivity of the medium at temperature A_0 [kJ/h m °C];
K_1, thermal conductivity of the medium per unit temperature [kJ/h m °C^2];
n, number of harmonics;
Re, real part of the quantity;
$T(x,t)$, temperature of the medium [°C];
t, time [h];
x, vertical axis.

Greek symbols

ρ, density of the medium [kg/m^3];
ω, 2π/period [h^{-1}];
$\theta(x,t)$, $= [T(x,t)-A_0]$ [°C];
α_0, absorption coefficient of the surface;
α, $= \left(\dfrac{\omega\rho c}{2K_0}\right)^{1/2}$;
ε, long wave emissivity of the surface;
σ_n, phase factor of nth harmonic of the surface temperature, Radian.

INTRODUCTION

The usual analyses [1–7] of periodic heat transfer are based on the assumption that the thermal conductivity of the medium is independent of temperature. In many cases of interest (e.g. some insulators [10] and materials with moisture [11]) this assumption is not valid.

In the present communication the authors have analysed the effect of temperature dependence of thermal conductivity on the temperature distribution with time and depth in a semi-infinite medium whose surface undergoes periodic variation of temperature. A linear temperature dependence of the thermal conductivity valid for most of materials of interest in a limited range of temperature has been assumed.

A periodic Fourier series solution of the one dimensional heat conduction equation satisfying the appropriate boundary conditions has been obtained in the perturbation approximation and an explicit expression for the temperature as a function of depth and time has been derived.

Daily and annual variation of temperature in ground at Kuwait has been numerically evaluated assuming the surface temperature to be equal to solair temperature.

ANALYSIS

We consider the temperature distribution $T(x,t)$ in a semi-infinite medium extending to infinity in the x-direction. The temperature at the surface $x = 0$ is assumed to be periodic and can be expanded in the form of a Fourier series as

$$T(x=0) = A_0 + Re \sum_{n=1}^{\infty} a_n \exp(in\omega t), \tag{1}$$

where $a_n = A_n \exp(-i\sigma_n)$; A_n and σ_n are the amplitude and phase factor respectively of the nth harmonic of the surface temperature.

The temperature distribution in the medium is governed by the one dimensional heat conduction equation [6]

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial \theta}{\partial x}\right) = K \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial K}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right)^2, \tag{2}$$

where

$$\theta(x,t) = T(x,t) - A_0,$$

and K is the thermal conductivity of the medium given by

$$K = K_0 + K_1\theta. \tag{3}$$

Substituting for K from equation (3) in equation (2) one obtains

$$\rho c \frac{\partial \theta}{\partial t} - K_0 \frac{\partial^2 \theta}{\partial x^2} = \frac{K_1}{2} \frac{\partial^2}{\partial x^2}(\theta^2). \tag{4}$$

To solve this equation we make use of the following appropriate boundary conditions:

$$\theta(x=0) = T(x=0) - A_0 = Re \sum_{n=1}^{\infty} a_n \exp(in\omega t), \tag{1A}$$

and $\theta(x \to \infty)$ is finite when $K_1/K_0 \ll 1$, the solution of equation (4) may, using the perturbation technique and the given boundary conditions, be written as

$$\theta(x,t) = Re \sum_{n=1}^{\infty} a_n \exp(-\beta_n x)\exp(in\omega t) + \frac{K_1}{K_0} F(x,t),$$

where

$$F(x,t) = \frac{1}{2}\Bigg[Re \sum_{n=1}^{\infty} \frac{a_n a_n^*}{2}\{1 - \exp[-(\beta_n + \beta_n^*)x]\}$$
$$+ Re \sum_{m=1}^{\infty} \frac{a_m a_{m+1}^*(\beta_m + \beta_{m+1}^*)^2}{[(\beta_m + \beta_{m+1}^*)^2 - \beta_1^2]}\{\exp(-\beta_1 x) - \exp[-(\beta_m + \beta_{m+1}^*)x]\}\exp(i\omega t)$$

$$+\left(\frac{2a_1^2\beta_1^2}{4\beta_1^2-\beta_2^2}[\exp(-\beta_2 x)-\exp(-2\beta_1 x)]\right.$$
$$\left.+\sum_{m=1}^{\infty}\frac{a_m a_{m+2}^*(\beta_m+\beta_{m+2}^*)^2}{[(\beta_m+\beta_{m+2}^*)^2-\beta_2^2]}\{\exp(-\beta_2 x)-\exp[-(\beta_m+\beta_{m+2}^*)x]\}\right)\exp(i2\omega t)$$
$$+\left(\frac{a_1 a_2(\beta_1+\beta_2)^2}{[(\beta_1+\beta_2)^2-\beta_3^2]}\{\exp(-\beta_3 x)-\exp[-(\beta_1+\beta_2)x]\}\right.$$
$$\left.+\sum_{m=1}^{\infty}\frac{a_m a_{m+3}^*(\beta_m+\beta_{m+3}^*)^2}{[(\beta_m+\beta_{m+3}^*)^2-\beta_3^2]}\{\exp(-\beta_3 x)-\exp[-(\beta_m+\beta_{m+3}^*)x]\}\right)\exp(i3\omega t)$$
$$\left.+\text{higher harmonics}\right]. \tag{6}$$

DISCUSSION OF RESULTS

The first term on the right hand side of equation (5) represents different harmonics, which propagate linearly, independent of each other. The second term, however represents the interaction of different harmonics; the terms, representing a given harmonic are dependent on the other harmonics. Specifically the amplitude and phase of a given harmonic at a point is a function, not alone of the amplitude and phase of that harmonic at $x = 0$ but also of the amplitude and phase of other harmonics at $x = 0$. This mixing of harmonics is not unique to the present case but is a characteristic feature of wave propagation in nonlinear media; the generation of harmonics of electromagnetic waves in nonlinear plasmas and semiconductors is very similar to that in the present case [8]. Thus it is seen that on account of the temperature dependent thermal activity the temperature distribution is shifted by an amount $(K_1/K_0)F(x,t)$ from the solution corresponding to thermal conductivity (K_0) at constant surface temperature (A_0).

To have a numerical appreciation of the results, we consider the temperature distribution in a semi-infinite (thickness much larger than characteristic length α^{-1}) ground or insulating material slab, whose horizontal surface is exposed to solar radiation and the atmospheric air. The surface characteristics have been assumed to correspond to absorption coefficient (α_0) of 0.9 for solar radiation, long wave emissivity (ε) equal to unity and convective heat transfer coefficient equal to 81.8 kJ/h m^2 °C (for a wind velocity of 12 km/h). The identity of solair temperature to the surface temperature is an excellent assumption for the annual variation, but not so good for the daily variation. This is not too serious because the aim of the present analysis is to study the effect of temperature dependence of thermal conductivity. The daily variation considered here corresponds to 21 March, 1975 in Kuwait (for $\alpha_0 = 0.9$ and $\varepsilon = 1.0$) while the annual variation is based on data averaged over 10 yr available from Kuwait International Airport records. The Fourier analyses of solair temperature for daily and annual variations [9] are given in Tables 1 and 2 respectively. Since a_1 and a_2 are much greater than $a_3, a_4 \ldots$ etc., we have only considered the first two harmonics in our calculations.

DAILY VARIATION

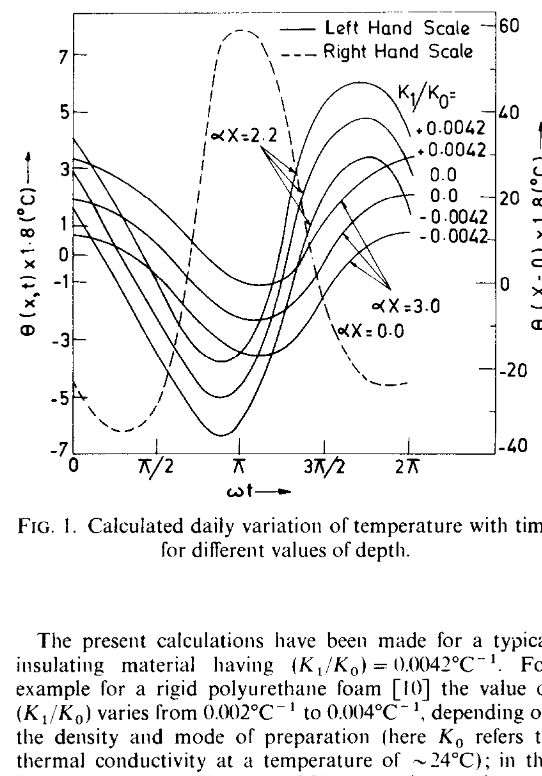

FIG. 1. Calculated daily variation of temperature with time for different values of depth.

The present calculations have been made for a typical insulating material having $(K_1/K_0) = 0.0042$°C^{-1}. For example for a rigid polyurethane foam [10] the value of (K_1/K_0) varies from 0.002°C^{-1} to 0.004°C^{-1}, depending on the density and mode of preparation (here K_0 refers to thermal conductivity at a temperature of ~24°C); in the low temperature regions (< 0°C) K_1/K_0 is negative and has a magnitude of the same order. In some other typical insulators or materials with moisture [11] even a higher value of (K_1/K_0) is expected.

For typical polyurethane foam, from available data of K, ρ and c [10–12]

$\alpha = 8.48\,\text{m}^{-1}$ for daily variation
$\alpha = 0.46\,\text{m}^{-1}$ for annual variation

Table 1. Fourier analysis of daily variation of solar temperature on ground in Kuwait (21 March, 1975)

n	0	1	2	3	4	5	6
A_n °C	28.5625	23.8535	10.2005	1.3760	1.4460	0.7940	0.3605
σ_n radians		3.40778	0.12662	2.78537	4.13958	1.44181	3.29172

Table 2. Fourier analysis of annual variation of solar temperature on ground in Kuwait

n	0	1	2	3	4	5	6
A_n °C	41.8055	18.4900	2.2205	2.3973	1.6230	0.7065	0.3398
σ_n radians		3.26334	2.38678	2.28844	3.79869	4.60673	3.14154

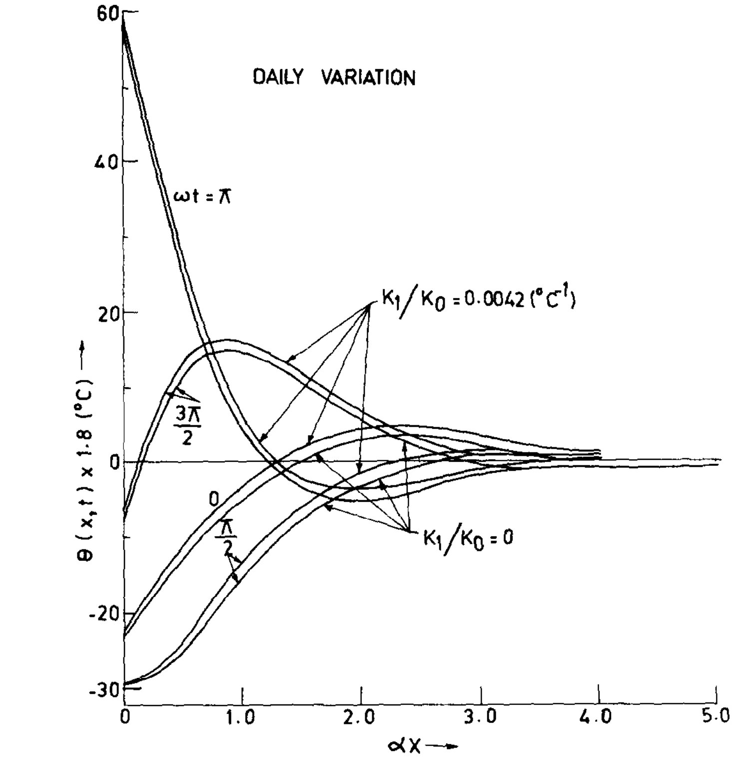

FIG. 2. Calculated daily variation of temperature with depth (αx) for different values of time.

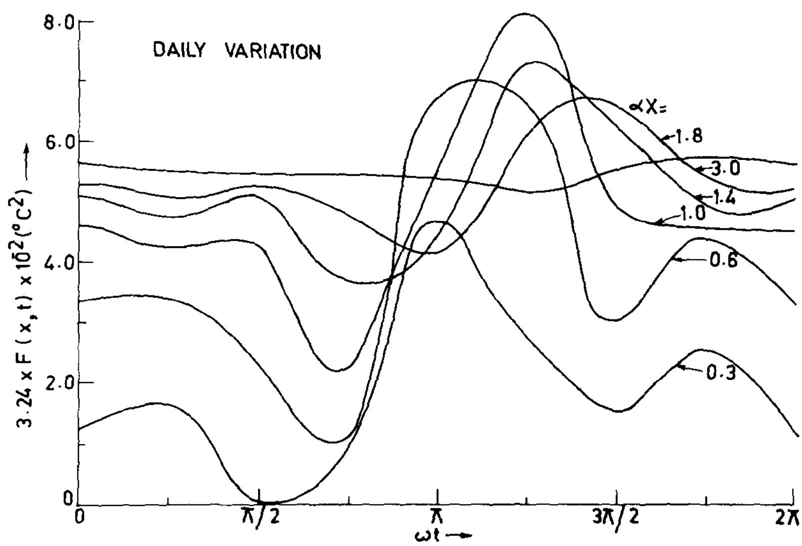

FIG. 3. Calculated daily variation of the function $F(x, t)$ with time ωt at various depths.

and for Kuwait soil [9]

$\alpha = 16.23\,\mathrm{m}^{-1}$ for daily variation
$\alpha = 0.85\,\mathrm{m}^{-1}$ for annual variation.

Figure 1 shows the daily variation of temperature with time (ωt) for different values of depth (αx) and K_1/K_0. It is seen that as the depth increases the effect of temperature dependence of thermal conductivity becomes increasingly significant. However, the peak values get reduced with increasing αx. The temperature is in general higher for $K_1/K_0 > 0$ and lower for $K_1/K_0 < 0$ as compared to the case when $K_1/K_0 = 0$; this is as expected.

Figure 2 shows the daily variation of temperature with depth (αx) for different values of time (ωt) and K_1/K_0. For $\omega t = \pi$, the temperature decreases rapidly with increasing αx and then increases but for $\omega t = 0$, $\pi/2$ and $3\pi/2$, it first increases and then decreases slowly. This is because of different phase lags introduced by the various harmonics.

The effect of temperature dependence of thermal conductivity on temperature distribution with distance and time can further be appreciated with reference to the function $F(x, t)$ defined in Section 2 [see equation (6)].

Figure 3 shows the daily variation of the function $F(x, t)$ with time (ωt) at various depths. It is found that $F(x, t)$ is a

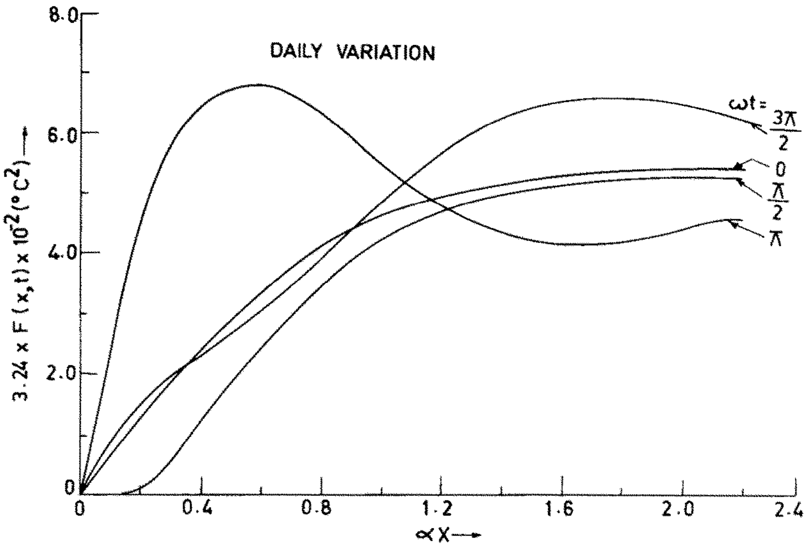

FIG. 4. Calculated daily variation of the function $F(x, t)$ with depth (αx) for various values of ωt.

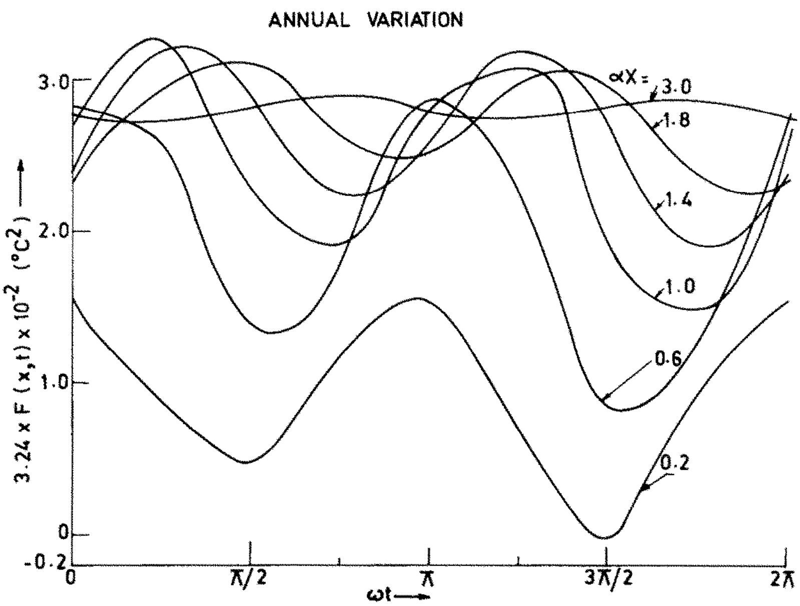

FIG. 5. Calculated annual variation of the function $F(x, t)$ with time ωt at various depths.

periodic function of time. The magnitude of $F(x, t)$ decreases with the increase of the depth (αx) and $F(x, t)$ is smoothened out at larger depths. This is because the amplitudes of various harmonics decay exponentially with depth. Consequently for depth greater than $\alpha x = 3$ only the time independent part of the temperature is predominant.

Figure 4 shows the daily variation of $F(x, t)$ with depth (αx) for various values of ωt. For $\omega t = 0$ and $\pi/2$ the function $F(x, t)$ increases with the increase of αx and for $\omega t = \pi$ and $3\pi/2$, the function $F(x, t)$ first increases and then decreases. This is because of the different phase lag introduced in various harmonics. It must be noted here that $F(x, t)$ is a decaying periodic function of αx.

For annual variation ($\omega = 2\pi$/y) the results are similar to those obtained for daily variation; only the coefficients a_n [see equation (1)] are different. Figures (4) and (5) present the variation of $F(x, t)$ with depth and time for the annual variation of the surface temperature.

CONCLUSION

The authors have presented an analysis for temperature distribution in a semi-infinite medium, having temperature dependent thermal conductivity and periodic surface temperature. Unlike the case of constant thermal conductivity, the propagation of a given harmonic is also affected by the amplitude and phase of the other harmonics; this also causes the time independent part of the temperature to be a function of depth. Since many building materials [10, 11] display a temperature dependent thermal conductivity, a nonlinear analysis has thus to be used in evaluation of temperature distribution and thermal flux, to get realistic results.

Acknowledgements—The authors are grateful to Dr. S. S. Mathur and Dr. R. P. Sharma for many stimulating discussions.

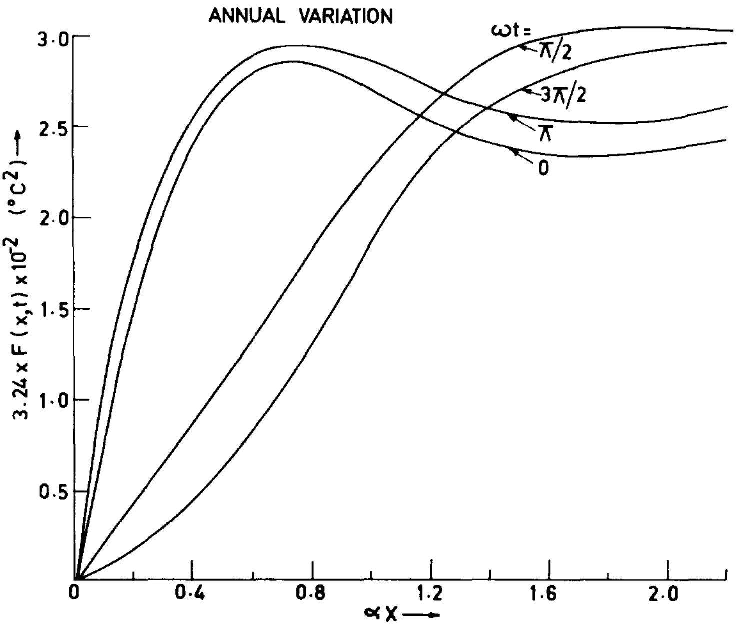

FIG. 6. Calculated annual variation of the function $F(x, t)$ with depth αx for various values of ωt.

REFERENCES

1. E. R. G. Eckert and R. M. Drake, *Heat and Mass Transfer*. McGraw-Hill, New York (1959).
2. J. K. Threlkeld, *Thermal Environmental Engineering*. Prentice-Hall, Englewood Cliffs, New Jersey (1970).
3. C. O. Mackey and L. T. Wright, Jr., Periodic heat flow composite walls or roofs, *Am. Soc. Heat. Vent. Engrs Trans.* **52**, 283–296 (1946).
4. J. P. Stewart, Solar heat gain through walls and roofs for cooling load calculations, *Am. Soc. Heat Vent. Engrs Trans.* **54**, 361–388 (1948).
5. Charles E. Bullock, Periodic heat transfer in walls and roofs, Plan B Master's Degree paper, University of Minnesota (1961).
6. H. S. Carslaw and J. C. Jaegar, *Conduction of Heat in Solids*. Oxford University Press, Oxford (1967).
7. F. J. Bayley, J. M. Owen and A. B. Turner, *Heat Transfer*. Thomas Nelson, London (1972).
8. M. S. Sodha and P. K. Kaw, Theory of harmonics in plasmas, *Adv. Electronics Electron Phys.* **27**, 187–293 (1969).
9. A. K. Khatry, M. S. Sodha and M. A. S. Malik, Periodic variation of ground temperature with depth and time, *Solar Energy* **20**, 425–427 (1978).
10. T. T. Healy, Polyurethane foams, in *Proceedings of the Symposium on Polyurethane Foams*, p. 125, London (1963).
11. ASHRAE Hand book of Fundamentals, p. 243. American Society of Heating, Refrigeration and Air-conditioning Engineers, New York (1967).
12. Encyclopedia of Polymer Science and Technology, Vol. 3, p. 99. Interscience, New York (1965).